AD-A246 883

Report No. NADC-91112-50

A PRIMER ON POLYNOMIAL RESULTANTS

Ronald F. Gleeson, Ph.D.
Department of Physics
Trenton State College
Trenton, NJ 08650

Robert M. Williams, Ph.D.
Mission Avionics Technology Department (Code 505A)
NAVAL AIR DEVELOPMENT CENTER
Warminster, PA 18974-5000

5 DECEMBER 1991

FINAL REPORT
Period Covering March 1991 to December 1991

Approved for Public Release; Distribution is Unlimited

Prepared for
OFFICE OF NAVAL RESEARCH
800 N. Quincy Street
Arlington, VA 22217

NOTICES

REPORT NUMBERING SYSTEM — The numbering of technical project reports issued by the Naval Air Development Center is arranged for specific identification purposes. Each number consists of the Center acronym, the calendar year in which the number was assigned, the sequence number of the report within the specific calendar year, and the official 2-digit correspondence code of the Command Officer or the Functional Department responsible for the report. For example: Report No. NADC-88020-60 indicates the twentieth Center report for the year 1988 and prepared by the Air Vehicle and Crew Systems Technology Department. The numerical codes are as follows:

CODE	OFFICE OR DEPARTMENT
00	Commander, Naval Air Development Center
01	Technical Director, Naval Air Development Center
05	Computer Department
10	AntiSubmarine Warfare Systems Department
20	Tactical Air Systems Department
30	Warfare Systems Analysis Department
40	Communication Navigation Technology Department
50	Mission Avionics Technology Department
60	Air Vehicle & Crew Systems Technology Department
70	Systems & Software Technology Department
80	Engineering Support Group
90	Test & Evaluation Group

PRODUCT ENDORSEMENT — The discussion or instructions concerning commercial products herein do not constitute an endorsement by the Government nor do they convey or imply the license or right to use such products.

Reviewed By: [illegible] Date: 1/7/92
Branch Head

Reviewed By: [illegible] Date: 1/8/92
Division Head

Reviewed By: [illegible] Date: 1/16/92
Director/Deputy Director

REPORT DOCUMENTATION PAGE

Form Approved
OMB No 0704-0188

... estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources ... and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this ... Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson ... and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503

1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	5 December 1991	Final - March 1991 - December 1991

4. TITLE AND SUBTITLE
A Primer On Polynomial Resultants

5. FUNDING NUMBERS

6. AUTHOR(S)

Ronald F. Gleeson*, Ph.D.
Robert M. Williams, Ph.D.

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Mission Avionics Technology Department (Code 505A)
NAVAL AIR DEVELOPMENT CENTER
Warminster, PA 18974-5000

8. PERFORMING ORGANIZATION REPORT NUMBER
NADC-91112-50

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
OFFICE OF NAVAL RESEARCH
800 N. Quincy Street
Arlington, VA 22217

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

*Ronald F. Gleeson
Department of Physics
Trenton State College, Trenton, NJ 08650

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for Public Release; Distribution is Unlimited

12b. DISTRIBUTION CODE

13. ABSTRACT (Maximum 200 words)

Nonlinearity is one of the most stubborn difficulties of contemporary engineering and science. In this paper we are concerned with a broadly useful tool, the resultant, for manipulating polynomial nonlinearities, and we review several techniques for solving systems of nonlinear polynomial equations. The resultant, a classical algebraic tool, has become much more practical recently with the advent of symbolic software (such as Mathematica and Maple) which can evaluate 10 X 10 symbolic determinants in a matter of minutes on a desktop computer.

While much of this paper is concerned with applying resultants to systems of univariate equations, the last section considers the generalization to the multivariate situation. Nonlinear multivariate applications appear in various areas of engineering such as chaos, signal processing, circuit theory, robotics and control theory. Two illustrations of the power of the resultant formalism are provided. First, the problem of finding the coordinates on the Earth's surface viewed by each pixel of a reconnaissance aircraft camera is discussed. Second, the Lorenz model of chaos theory is considered.

14. SUBJECT TERMS
Resultant, Nonlinear Equations, Mathematica, Reconnaissance, Symbolic Computing

15. NUMBER OF PAGES

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
Unclassifed	Unclassified	Unclassified	UL

TABLE OF CONTENTS

Section	Page
List of Figures	ii
List of Tables	ii
Abstract	iii
1. Introduction	1
A. Univariate Polynomial Systems of Equations	1
B. Multivariate Polynomial Systems of Equations	2
II. Existence and Number of Common Roots for Univariate Equations	2
A. Resultant Method	2
B. Rank Method	2
III. Values of the Common Roots for Univariate Equations	3
A. One Common Root	3
1. Derivatives of Resultants	3
2. Cramer's Rule	5
B. Two Common Roots	5
1. Derivatives of Resultants	5
2. Greatest Common Divisor Using Subresultants	6
IV. Repeated Roots of a Single Univariate Equation	8
V. Multivariate Generalization	10
A. Reconnaissance Problem	10
B. Chaos Problem	13
VI. Conclusion	14
Acknowledgments	15
Appendix A: Sample Run Sessions	16
Appendix B: Software Timing Survey	18
References	20

LIST OF FIGURES

No.	Figure	Page
1	Sphere Intercepted By Pixel Line Of Sight	13

LIST OF TABLES

No.	Table	Page
1	Execution Time	19

Abstract

Nonlinearity is one of the most stubborn difficulties of contemporary engineering and science. In this paper we are concerned with a broadly useful tool, the resultant, for manipulating polynomial nonlinearities, and we review several techniques for solving systems of nonlinear polynomial equations. The resultant, a classical algebraic tool, has become much more practical recently with the advent of symbolic software (such as Mathematica and Maple) which can evaluate 10 X 10 symbolic determinants in a matter of minutes on a desktop computer.

While much of this paper is concerned with applying resultants to systems of univariate equations, the last section considers the generalization to the multivariate situation. Nonlinear multivariate applications appear in various areas of engineering such as chaos, signal processing, circuit theory, robotics and control theory. Two illustrations of the power of the resultant formalism are provided. First, the problem of finding the coordinates on the Earth's surface viewed by each pixel of a reconnaissance aircraft camera is discussed. Second, the Lorenz model of chaos theory is considered.

I. Introduction

A. Univariate Polynomial Systems of Equations

Given two univariate polynomial equations such as

$$d\,x^3 + c\,x^2 + b\,x + a = 0 \quad \text{and} \tag{1.1}$$
$$C\,x^2 + B\,x + A = 0 \tag{1.2}$$

there are a variety of methods available for finding any common roots. One way, obviously, is to use some brute force technique to numerically determine all the roots of both equations (e.g. Newton's Root Finder Method). The common roots can then be selected as the intersection of these two sets.

This paper describes a more elegant method for solving this problem which utilizes the resultant [1] [2] (defined below). In this section we will define the key elements involved in these techniques; in later sections, we will describe how these elements can be used to solve polynomial equations.

One way to obtain the resultant is by taking the determinant of the Sylvester matrix. The Sylvester matrix is formed by padding zeroes before and/or after the coefficients of the two polynomials For the polynomials in equations (1.1) and (1.2) above, the Sylvester matrix would be:

$$\begin{vmatrix} d & c & b & a & 0 \\ 0 & d & c & b & a \\ 0 & 0 & C & B & A \\ 0 & C & B & A & 0 \\ C & B & A & 0 & 0 \end{vmatrix}$$

The size of the matrix is equal to the sum of the degrees of the highest order terms of each of the polynomials. For example, for the above two polynomials, the highest order term in the first polynomial is three, and in the second polynomial it is two. Therefore, the matrix is 5 X 5.

The subresultant is formed from the determinant of the Inner matrix. The Inner matrix is formed by removing the outer layer of elements from the Sylvester matrix (the top and bottom rows, and the first and last columns). For example, for the above case we would have

$$\begin{vmatrix} d & c & b \\ 0 & C & B \\ C & B & A \end{vmatrix}$$

The 2nd subresultant, 3rd subresultant, etc., can be formed in the same manner. The 2nd subresultant in the above example would be the single element $|C|$.

B. Multivariate Polynomial Systems of Equations

Given a pair of multivariate equations such as

$$f\,x\,y^2 + d\,x^2\,y^2 + c\,x^3\,y^2 + b\,x\,y + a\,y = 0 \qquad (1.3)$$
$$D\,x^2\,y^2 + C\,x^3\,y + B\,x\,y + A\,x^3 = 0 \qquad (1.4)$$

the resultant methodology is still useful. The resultant applied to this pair of polynomial equations will reduce the number of variables by one. The equations are treated in much the same way as with the single variable case. The main difference is that now the coefficients of the variable to be eliminated (for example, y) are no longer constants, but rather are polynomials in the remaining set of variables (for the equations above, the coefficients of y are polynomials in x). For examples using nonlinear multivariate equations, see Section V, below.

II. Existence and Number of Common Roots for Univariate Equations

A. Resultant Method

In general, the number of common roots for two polynomials can be found by checking the values of the resultant and the subresultants If the resultant is nonzero, then there are no common roots; whereas, if it is zero, there is at least one common root [3] [4]. Then the subresultants must be checked in increasing order (decreasing determinant size) for a nonzero result. If the Nth subresultant is the first subresultant to yield a nonzero result, then there are N common roots.

This method lends itself to a symbolic solution because it only involves rational combinations of the coefficients of the polynomials in contrast to nonrational iterative processes such as root finders.

B. Rank Method

The rank of the matrix can be obtained in a variety of ways. One way is to determine the singular values. The rank is the number of nonzero singular values.

The rank of the Sylvester matrix can be used to determine the number of common roots. If the rank of the Sylvester matrix is equal to its size (dimension), then there are no common roots. However, if the rank is less than the size, then the difference between the size and the rank is the number of common roots. For example, if the Sylvester matrix is 5 X 5 and the rank is 3, then the number of common roots is 5 - 3 = 2.

In contrast to the resultant method discussed above, this method does not lend itself to a symbolic solution. A numerical algorithmic approach is typically utilized to solve for the singular values.

III. Values of the Common Roots for Univariate Equations

A. One common root

1. Derivatives of Resultants

The value of the common root (cr) can be determined by computing the ratio of the first derivatives of the resultant (R) with respect to two successive coefficients [5]. Expressed as a formula,

$$cr = \frac{dR/dC}{dR/dB} \tag{3.1}$$

where C and B are any two successive coefficients in either polynomial (C is the coefficient of the higher order term).

To illustrate this procedure, let us consider the two equations in the Introduction, namely

$$d\,x^3 + c\,x^2 + b\,x + a = 0 \quad \text{and} \tag{3.2}$$
$$C\,x^2 + B\,x + A = 0 \tag{3,3}$$

Recall, that the Sylvester matrix of these equations was

$$\begin{vmatrix} d & c & b & a & 0 \\ 0 & d & c & b & a \\ 0 & 0 & C & B & A \\ 0 & C & B & A & 0 \\ C & B & A & 0 & 0 \end{vmatrix}$$

The resultant, R, is the determinant of this matrix, and this turns out to be

$$\begin{aligned} R = \; & -C^3 a^2 + B\,C^2 a\,b - A\,C^2 b^2 - B^2 C\,a\,c + 2\,A\,C^2 a\,c + A\,B\,C\,b\,c \\ & - A^2 C\,c^2 + B^3 a\,d - 3\,A\,B\,C\,a\,d - A\,B^2 b\,d + 2\,A^2 C\,b\,d \\ & + A^2 B\,c\,d - A^3 d^2 \end{aligned} \tag{3.4}$$

The first derivatives of R with respect to C and B are

$$\begin{aligned} dR/dC = \; & -3\,C^2 a^2 + 2\,B\,C\,a\,b - 2\,A\,C\,b^2 - B^2 a\,c + 4\,A\,C\,a\,c \\ & + A\,B\,b\,c - A^2 c^2 - 3\,A\,B\,a\,d + 2\,A^2 b\,d \end{aligned} \tag{3.5}$$

$$\begin{aligned} dR/dB = \; & C^2 a\,b - 2\,B\,C\,a\,c + A\,C\,b\,c + 3\,B^2 a\,d - 3\,A\,C\,a\,d \\ & - 2\,A\,B\,b\,d + A^2 c\,d. \end{aligned} \tag{3.6}$$

Suppose, for example, that the two equations were

$$f = (x - 3)(x - 1)(x - 2) = x^3 - 6x^2 + 11x - 6 \tag{3.7}$$
$$g = (x - 3)(x - 5) = x^2 - 8x + 15 \tag{3.8}$$

If the reader were presented with these two equations (3.7) and (3.8) in the unfactored form and were unaware that the common root is 3, he could plug $d = 1$, $c = -6$, $b = 11$, $a = -6$, $C = 1$, $B = -8$, and $A = 15$. into equations (3.5) and (3.6). The derivatives evaluate to $dR/dC = -216$ and $dR/dB = -72$. The ratio of these two derivatives, equation (3.1), yields the common root to be 3, as expected.

2. Cramer's Rule

Another method for solving for a single common root involves Cramer's Rule [6]. First, the bottom row is dropped from the Sylvester matrix. Second, two determinants are formed: one by dropping the next to last column (which we shall call "Next-to-Last"); the other, by dropping the last column (which we shall call "Last").

Using, again, the example discussed in the Introduction, we have

Next-to-Last = Determinant of the following matrix: (3.9)

$$\begin{vmatrix} d & c & b & 0 \\ 0 & d & c & a \\ 0 & 0 & C & A \\ 0 & C & B & 0 \end{vmatrix}$$

Last = Determinant of the following matrix (3.10)

$$\begin{vmatrix} d & c & b & a \\ 0 & d & c & b \\ 0 & 0 & C & B \\ 0 & C & B & A \end{vmatrix}$$

The single common root (cr) is then the negative of the ratio of the Next-to-Last determinant to the Last determinant. Expressed as a formula, we have

$$cr = -\frac{\text{Next-to-Last}}{\text{Last}} \qquad (3.11)$$

If one were trying to solve equations (3.7) and (3.8) using this approach, Next-to-Last and Last evaluate to 36 and -12 respectively, and the negative of their ratio (the common root) is 3.

B. Two Common Roots

1. Derivatives of Resultants

If there are two common roots, the second derivatives of the resultant with respect to two successive coefficients can be used

to solve for their values [7]. These second derivatives are inserted into a quadratic equation, and the solutions to the quadradic equation are the two common roots.

Suppose C and B are two successive coefficients in either polynomial, and R is the resultant. Let us define

$$u = d^2R/dC^2 \tag{3.12}$$
$$v = d^2R/dCdB \tag{3.13}$$
$$w = d^2R/dB^2 \tag{3.14}$$

The two common roots are then obtained from the following:.

$$cr = \frac{v \pm \sqrt{v^2 - u\,w}}{w} \tag{3.15}$$

We will apply this formalism to equations (1.1) and (1.2), since we already know the first derivatives of the resultant from equations (3.5) and (3.6). Taking the second derivatives, we get symbolically

$$u = d^2R/dC^2 = -6\ C\ a^2 + 2\ B\ a\ b - 2\ A\ b^2 + 4\ A\ a\ c \tag{3.16}$$
$$v = d^2R/dCdB = 2\ C\ a\ b - 2\ B\ a\ c + A\ b\ c - 3\ A\ a\ d \tag{3.17}$$
$$w = d^2R/dB^2 = -2\ C\ a\ c + 6\ B\ a\ d - 2\ A\ b\ d \tag{3.18}$$

A numerical example of this approach can readily be had by altering equations (3.7) and (3.8) so that they have two common roots, 3 and 5. For example, suppose

$$f = (x - 3)(x - 5)(x - 2) = x^3 - 10x^2 + 31x - 30 \tag{3.19}$$
$$g = (x - 3)(x - 5) = x^2 - 8x + 15 \tag{3.20}$$

If we substitute $d = 1$, $c = -10$, $b = 31$, $a = -30$, and $C = 1$, $B = -8$, and $A = 15$ into equations (3.16), (3.17), and (3.18) we get $u = -1350$, $v = -360$, and $w = -90$. If these values are then inserted into equation (3.15), the common roots turn out to be 3 and 5.

2. Greatest Common Divisor (GCD) Using Subresultants

A second method for finding the values when there are two common roots involves finding the polynomial GCD using the first nonzero subresultant [8] [9]. The roots obtained by setting this

GCD equal to zero are the common roots of the two given polynomials.

The technique involves replacing the last column of the first nonzero subresultant (we will call this "FNS") by a series of elements formed by multiplying the two given polynomials by different powers of the unknown (e.g. x). The determinant of this altered subresultant is the GCD.

Suppose the coefficients from the first given polynomial are contributing "i" rows to the FNS, and the second polynomial's coefficients are contributing "j" rows. The replacement column then has "i+j" elements. The top element in the replacement column is formed by multiplying the first polynomial by "x" raised to the "i-1" power. The second element is had by multiplying the first polynomial by "x" raised to the "i-2" power. This continues until the element equals the first polynomial multiplied by one. The next element in the column is the second polynomial. This is followed by elements formed by the second polynomial multiplied by increasing powers of "x" until the power equals "j-1".

To illustrate this technique we would like to use again the equations (3.19) and (3.20). However, this case is too simple and does not demonstrate the technique very well. Recall that the Sylvester matrix is 5 X 5. Since we have two common roots, the first nonzero subresultant will be the second subresultant which is 1 X 1, a single element. Since this element is one of the coefficients from the equation (3.20), the altered subresultant (the GCD) ends up having one element which is

$$x^2 - 8x + 15 \tag{3.21}$$

When this GCD is set equal to zero, the roots are, of course, 3 and 5.

To obtain a better example we consider below another pair of equations (of higher order) which also have common roots 3 and 5.

$$\begin{aligned} f &= 2(x - 3)(x - 5)(x - 2)(x - 1) \\ &= 2x^4 - 22x^3 + 82x^2 - 122x + 60 \qquad (3.22) \\ g &= 3(x - 3)(x - 5)(x - 7) \\ &= 3x^3 - 45x^2 + 213x - 315 \qquad (3.23) \end{aligned}$$

The Sylvester matrix of these coefficients is

$$\begin{vmatrix} +2 & -22 & +82 & -122 & +60 & +0 & +0 \\ +0 & +2 & -22 & +82 & -122 & +60 & +0 \\ +0 & +0 & +2 & -22 & +82 & -122 & +60 \\ +0 & +0 & +0 & +3 & -45 & +213 & -315 \\ +0 & +0 & +3 & -45 & +213 & -315 & +0 \\ +0 & +3 & -45 & +213 & -315 & +0 & +0 \\ +3 & -45 & +213 & -315 & +0 & +0 & +0 \end{vmatrix}$$

The determinant of this matrix (the resultant) evaluates to zero, and the first subresultant is also zero. Since the second subresultant is not zero, there must be two common roots.

The second subresultant is

$$\begin{vmatrix} +2 & -22 & +82 \\ +0 & +3 & -45 \\ +3 & -45 & +213 \end{vmatrix}$$

The second subresultant modified by replacing the last column in the manner described above is

$$\begin{vmatrix} +2 & -22 & (2x^4-22x^3+82x^2-122x+60) \\ +0 & +3 & (3x^3-45x^2+213x-315) \\ +3 & -45 & (3x^4-45x^3+213x^2-315x) \end{vmatrix}$$

Suprisingly, the determinant of this matrix reduces down to $-540(x - 3)(x - 5)$. This is the GCD. The roots of this GCD set equal to zero are x equals 3 and 5 which are the two common roots of the two given polynomials.

IV. Repeated Roots of a Single Univariate Equation: Discriminant

The resultant methodology is also very useful for determining whether a single univariate polynomial equation has repeated roots. One test for repeated roots is to check if the discriminant (D) is zero. The reader will recall that for the quadratic equation the solutions are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \qquad (4.1)$$

In this case,

$$D = b^2 - 4a\ c \tag{4.2}$$

Obviously, when D = 0 there is a repeated root, namely $x = -b/2a$. It can easily be shown that an alternative way to express D for the quadratic case is

$$D = a^2 * (x_1 - x_2)^2 \tag{4.3}$$

where x_1 and x_2 are the two roots. In general, for a polynomial of order n, the discriminant is defined as

$$D = a_n^{2(n-1)}(x_1 - x_2)^2 \ldots (x_1 - x_n)^2 (x_2 - x_3)^2 \ldots (x_2 - x_n)^2 \ldots (x_{n-1} - x_n)^2 \tag{4.4}$$

where a_n is the coefficient of the highest order term, and $x_1, x_2, \ldots x_n$ are the roots. If any of the roots are repeated, D = 0. The problem is that since we are assuming that the roots are unknown, it is impossible to evaluate D directly from equation (4.4). However, the discriminant can be shown to be directly proportional to the resultant of the given polynomial and its first derivative [4]. This implies that if the resultant of a polynomial and its first derivative is zero, then there are repeated roots. Furthermore, this resultant is fairly straightforward to evaluate.

Consider, for example, the equation

$$f = (x - 3)^2 (x - 7) = x^3 - 13x^2 + 51x - 63 \tag{4.5}$$

The first derivative of f is

$$g = df/dx = 3x^2 - 26x + 51 \tag{4.6}$$

These two equations are in the form of (1.1) and (1.2). When the resultant is taken, it is found to be zero. Since this resultant of the function and its first derivative is zero, the discriminant must be zero, and this implies that there are repeated roots.

To find out the value of the repeated root, one can use the method discussed in section III(B.1) which the reader will recall utilizes the second derivatives of the resultant.

V. Multivariate Generalization

It was mentioned in the Introduction that if the system of polynomial equations involves more than one variable, the resultant technique can be used to reduce the number of variables. If there are sufficient equations, applying this resultant technique to pairs of equations can lead to unique solutions for the variables. This is often a nontrivial task when the equations are nonlinear. The resultant provides an organized approach to accomplish this task.

A. Reconnaissance Problem

In modern airborne reconnaissance it is important to be able to map the surface of the Earth onto the pixel plane of the camera. In the example which follows the coordinate system used has its origin at the center of the Earth. The aircraft is assumed to be on the positive z axis at coordinate z_0 The x and y axes point East and North respectively from the aircraft. We will assume that we have earlier computed the direction cosines (cos-x, cos-y, cos-z) of the vector pointing from a given pixel toward the Earth. The Earth is modeled as a sphere of radius R. The successive use of resultants on pairs of equations is shown below to solve the problem of finding the point on the Earth's surface intercepted by the line of sight of each pixel.

The equations which characterize the line of sight of the pixel are:

$$\frac{x}{z - z_0} = \frac{\text{cos-x}}{\text{cos-z}} \tag{5.1}$$

$$\frac{y}{z - z_0} = \frac{\text{cos-y}}{\text{cos-z}} \tag{5.2}$$

The equation of the sphere is

$$x^2 + y^2 + z^2 = R^2 \tag{5.3}$$

Equations (5.1) , (5.2), and (5.3) are expressed below as (5.4), (5.5), and (5.6) respectively:

$$x - a\,z + b = 0 \tag{5.4}$$

$$y - c\,z + d = 0 \tag{5.5}$$

$$x^2 + y^2 + z^2 - R^2 = 0 \tag{5.6}$$

where the parameters a, b, c, and d are defined as:

$$a = \frac{\cos\text{-}x}{\cos\text{-}z} \qquad b = \left(\frac{\cos\text{-}x}{\cos\text{-}z}\right) z_0 \qquad c = \frac{\cos\text{-}y}{\cos\text{-}z} \qquad d = \left(\frac{\cos\text{-}y}{\cos\text{-}z}\right) z_0 \tag{5.7}$$

A Sylvester determinant can be formed from the coefficients of z in equations (5.4) and (5.5).

$$\begin{vmatrix} -a & (x+b) \\ -c & (y+d) \end{vmatrix}$$

When this determinant is set equal to zero, we obtain the following equation in x and y:

$$a\,y + (a\,d - c\,x - c\,b) = 0 \tag{5.8}$$

A second Sylvester determinant can be formed using the coefficients of z in (5.4) and (5.6).

$$\begin{vmatrix} -a & (x+b) & 0 \\ 0 & -a & (x+b) \\ 1 & 0 & (x^2+y^2-R^2) \end{vmatrix}$$

When this second determinant is set equal to zero, we obtain another equation in x and y:

$$a^2\,y^2 + (b^2 - a^2\,R^2 + 2\,b\,x + x^2 + a^2\,x^2) = 0 \tag{5.9}$$

Finally, a third Sylvester determinant can be formed using the coefficients of y in (5.8) and (5.9).

$$\begin{vmatrix} a & (a\,d - c\,x - c\,b) & 0 \\ 0 & a & (a\,d - c\,x - c\,b) \\ a^2 & 0 & (b^2 - a^2\,R^2 + 2\,b\,x + x^2 + a^2\,x^2) \end{vmatrix}$$

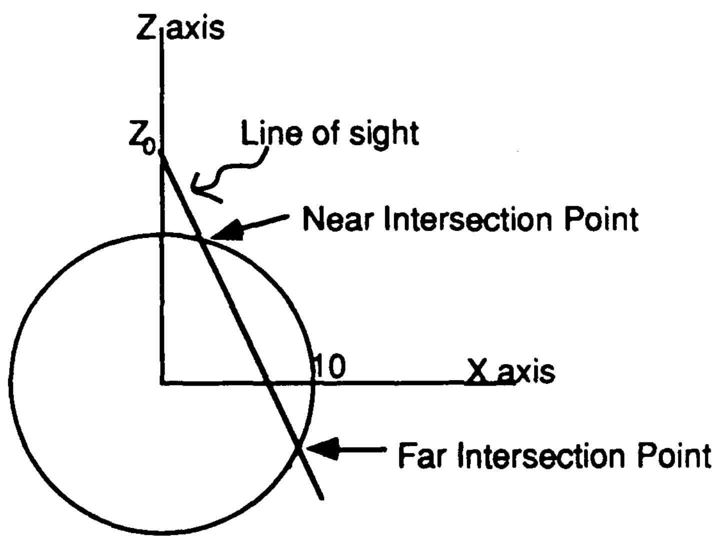
Setting this determinat equal to zero yields a quadratic equation in x:

$$(1 + a^2 + c^2)\, x^2 + (2\, b + 2\, b\, c^2 - 2\, a\, c\, d)\, x + (b^2 + b^2 c^2 - 2\, a\, b\, c\, d + a^2 d^2 - a^2 R^2) = 0 \qquad (5.10)$$

The quadratic formula is next applied to equation (5.10) to obtain the x roots. The smaller root is the x value of the intercept of the pixel's line of sight with the Earth. Obviously, if the roots are complex, the line of sight missed the Earth. The y value of the interception point can be obtained from equation (5.8), and the z value from equation (5.4).

To illustrate this technique, suppose that $R = 10$, $z_0 = 15$, and the angles of the pixel's line of sight with respect to the x, y, and z axes are 60°, 90°, and 150° respectively (it is in the x-z plane).

Figure 1: Sphere Intercepted By Pixel Line Of Sight



In this example, cos-x = cos(60°) = 0.5; cos-y = cos(90°) = 0; and cos-z = cos(150°) = -0.866. Using the definitions for parameters a, b, c and d (5.7) we have:

$$a = \frac{\text{cos-x}}{\text{cos-z}} = \frac{0.5}{-0.866} = -0.577 \qquad (5.11)$$

$$b = \left(\frac{\text{cos-x}}{\text{cos-z}}\right) z_0 = \left(\frac{0.5}{-0.866}\right) 15 = -8.66 \qquad (5.12)$$

$$c = \frac{\text{cos-y}}{\text{cos-z}} = \frac{0}{-0.866} = 0 \qquad (5.13)$$

$$d = \left(\frac{\cos\text{-}y}{\cos\text{-}z}\right) z_0 = \left(\frac{0}{-0.866}\right) 15 = 0 \tag{5,14}$$

Plugging these values for a, b, c, and d into equation (5.10) along with R = 10 we get:

$$1.33\ x^2 - 17.31\ x + 41.62 = 0 \tag{5.15}$$

Using the quadratic formula we get x = 3.19 and x = 9.80. The smaller value is the x coordinate of the line of sight intercept.

The value of y is had by putting x = 3.19 into equation (5.8) along with the values of the other parameters.

$$-0.577\ y + \ (-0.577\ )(0) - (0)\ (3.19) - (0)\ (-8.66) = 0 \tag{5.16}$$

Solving for y we get y = 0. This is expected since our line of sight is in the x-z plane

The z coordinate of the intercept is had by plugging the x value along with the values of the parameters into (5.4).

$$3.19 - (-0.577)\ z + (-8.66) = 0 \tag{5,17}$$

The z intercept turns out to be 9.48.

B. Chaos Problem

A common model in the theory of chaos, the Lorenz model, involves the following set of equations [10]:

$$dx_1/dt = -\ a\ x_1 + a\ x_2 \tag{5.18}$$
$$dx_2/dt = c\ x_1 - x_2 - x_1\ x_3 \tag{5.19}$$
$$dx_3/dt = -b\ x_3 + x_1\ x_2 \tag{5.20}$$

in which a, b, and c are positive constants.

In the limit when the derivatives are all zero, equation (5.18) reduces to simply $x_1 = x_2$. Replacing x_2 by x_1, in the other two equations (5.19 and 5.20). we have

$$0 = c\ x_1 - x_1 - x_1\ x_3 \tag{5.21}$$
$$0 = -b\ x_3 + (x_1)^2 \tag{5.22}$$

This is an especially simple set of equations. It was chosen for two reasons: (1) it allows the reader to see by inspection what the solution must be, that is $x_3 = c - 1$ and $x_1 = \mathrm{Sqrt}[b\,(c - 1)]$; (2) it allows the reader to easily evaluate the Sylvester determinant.

The resultant formalism can be shown to produce the same result. The Sylvester matrix is formed from the coefficients of the different powers of one of the variables. If the Sylvester matrix elements are, for example, the coefficients of x_1, then x_1 will not appear in this matrix and is said to be eliminated. In this case the coefficients of x_1, and therefore the elements of the Sylvester matrix, will involve polynomials in x_3. The reader can easily prove that the Sylvester matrix takes the form

$$\begin{vmatrix} c-1-x_3 & 0 & 0 \\ 0 & c-1-x_3 & 0 \\ 1 & 0 & -bx_3 \end{vmatrix}$$

Setting the determinant of this matrix equal to zero we have

$$(c - 1 - x_3)^2\,(-b\ x_3) = 0 \tag{5.23}$$

The nontrivial solution of (5.23) is $x_3 = c - 1$.

This problem was solvable without the Sylvester formalism. In other more complicated pairs of polynomial equations, it is not usually apparent how to algebraically eliminate one of the variables. However, the resultant formalism affords the user a systematic process for achieving this goal.

If there are several equations in as many unknowns, the user can apply the resultant formalism to pairs of equations, each time reducing the number of variables by one until there is eventually only one equation in one unknown [11] This single equation may have a high order, but a method such as the Newton Root Finder Method can then by used.

VI. Conclusion

The resultant is a useful tool for determining whether there are any common roots for a given pair of univariate polynomial equations. If common roots exist, the resultant can also be used to determine the values of these roots without determining all the roots of both polynomials. As a practical matter, these methods work very

well for one or two common roots. The common roots flow from an equation whose order is equal to the number of common roots. The solution of this equation becomes increasingly difficult as the number of common roots increases.

Since the resultant is proportional to the discriminant of a single univariate polynomial equation, it can provide a quick check for repeated roots.

When used on pairs of multivariate polynomial equations, each application of the resultant formalism provides an organized method for reducing the number of variables by one.

Acknowledgments

We would like to express our appreciation to David Saunders, Colin R. Williams, Mary Nicole Nazzaro, and Lisa Reno for their helpful comments on this paper.

APPENDIX A

SAMPLE RUN SESSIONS

Below we have included sample sessions which illustrate the syntax for evaluating the resultant on three different symbolic manipulation packages: Maple (version 4.2), Mathematica (version 1.2), and Cocoa (version 0.99b). Our hardware consisted of a Macintosh IIx equipped with 8 Meg of RAM.

Maple:

On Maple there are three optional methods for evaluating the resultant of two polynomials The first method is the simplest and fastest; it simply involves invoking the "resultant" command. The second method involves entering the Sylvester matrix (it can also be obtained by applying the "sylvester" command to the two polynomials), and then evaluating the determinant of this matrix using the "det" command The third method uses the "bezout" command; the "det" command is then applied to the output of the "bezout" command to produce the resultant.

Method 1:

```
f := d*x^3 + c*x^2 + b*x + a;
g := C*x^2 + B*x + A;
with(linalg);
resultant(f,g,x);
```

Method 2:

```
M := array(1..5,1..5,     [[d,c,b,a,0],
                           [0,d,c,b,a],
                           [0,0,C,B,A],
                           [0,C,B,A,0],
                           [C,B,A,0,0]]);
with(linalg);
det( M,sparse );
```

Method 3:

```
f:= d*x^3 + c*x^2 + b*x + a;
g:= C*x^2 + B*x + A;
with(linalg);
MapleBezArray:= bezout(f,g,x);
det(MapleBezArray);
```

Mathematica:

The first two methods described above are available to Mathematica users. There are two resultant commands: "Resultant" and "Resultant2". The latter is much faster. Also, Mathematica has not implemented direct commands equivalent to Maple's "sylvester" and "bezout".

Method 1:

```
f := d x^3 + c x^2 + b x + a
g := C x^2 + B x + A
Resultant2[f,g,x]
```

Method 2:

```
M := {{d,c,b,a,0},
      {0,d,c,b,a},
      {0,0,C,B,A},
      {0,C,B,A,0},
      [C,B,A,0,0}}
Det[M]
```

Cocoa:

Before the user can use Cocoa he must declare all his variables. This is done by using the mouse to pull down "Ring". After selecting "Set Ring", a screen appears which allows the user to declare variables. In our example, the user would type in the Variables window "abcdABCx" (without the quotes). Exit this screen by clicking "OK". Now, the user is ready to type in the appropriate input statements listed below.

Method 1:

$f = dx^3 + cx^2 + bx + a$

$g = Cx^2 + Bx + A$

Resultant(f,g,x)

Method 2:

```
M = Matrix(5,5,d,c,b,a,0,
                0,d,c,b,a,
                0,0,C,B,A,
                0,C,B,A,0,
                C,B,A,0,0)
Det(M)
```

APPENDIX B

SOFTWARE TIMING SURVEY

We timed how long various commands took to execute using the three symbolic manipulation packages listed in Appendix A (namely, Maple, Mathematica, and Cocoa) on our Macintosh IIx (eight Meg of RAM). We benchmarked the Resultant and Determinant commands on all three packages. The Determinant command was applied to the Sylvester matrix and therefore also determined the resultant. On Maple, recall from Appendix A, there is a third approach available to determine the resultant, namely the Bezout command. The Bezout command produces the Bezout matrix, and then the Determinant command can be applied to the Bezout matrix to yield the resultant.

In the table on the next page, the column under the heading "6=3+3" contains the execution times for determining the resultant of two univariate third order (cubic) equations using the commands discussed above. The Sylvester Determinants were 6x6. The "7=4+3" column contains the times for the resultant determination of a quartic and a cubic *equation, etc.*

Generally, the various commands were terminated after ten minutes if a result had not been determined; however, in some instances the system aborted the calculation due to lack of memory.

TABLE 1

EXECUTION TIME (Seconds)

Software Package	Size ** / Command	6=3+3	7=4+3	8=4+4	9=5+4	10=5+5	11=6+5	12=6+6	13=7+6
Maple									
	Resultant	4.2	4.8	8.2	17.8	54.0	293.5	*	*
	Determinant (Sylvester)	3.4	7.6	35.2	91.2	260.5	*	*	*
	Bezout and Determinant	2.3	3.3	10.2	28.2	85.7	359.0	*	*
Mathematica									
	Resultant2	1.1	1.6	2.7	8.1	19.3	97.6	266.4	*
	Determinant (Sylvester)	1.5	7.3	14.0	48.9	183.3	*	*	*
Cocoa									
	Resultant	7.1	*	*	*	*	*	*	*
	Determinant (Sylvester)	5.8	47.2	*	*	*	*	*	*

* = Execution took at least ten minutes or was aborted by the system due to lack of memory.

** = Sizes are presented as a set of three numbers, for example, 7 = 4 + 3. The first number (e.g.,7) is the size of the Sylvester matrix. The second and third numbers (e.g.,4 and 3) are the orders of the two polynomials.

References

[1] Gerber, H.U., "Wronski's Formula and the Resultant of Two Polynomials", The American Mathematical Monthly, Vol. 91, pp. 644-646, (December, 1984).

[2] Griffiths, H.R., "Calley's Version of the Resultant of Two Polynomials", The American Mathematical Monthly. Vol. 88, pp. 328-338, (May, 1981).

[3] Hall, H.S., and Knight, S.R. Higher Algebra, pp. 446 to 449, Fourth Edition, MacMillan and Co. Ltd, London, 1927.

[4] Mostowski, A. and Stark M., Introduction To Higher Algebra pp. 371 to 394, Pergamon Press, New York, 1964.

[5] Meyler, D.S., and Sutton, O.G., A Compendium of Mathematics and Physics, p. 17, English Universities Press Ltd, London, c.1958.

[6] Sederberg, T.W., and Goldman, R.N., "Algebraic Geometry for Computer-Aided Geometric Design", IEEE CG&A, June, 1986.

[7] Salmon, G. Lessons Introductory To The Modern Higher Algebra, pp. 90 to 98, Chelsea Publishing Company, Bronx, NY, 1964.

[8] Jury,E.I.,Inners and Stability of Dynamic Systems, pp.1 to 20, Robert E. Krieger Publishing company, Malabar, Florida, 1982.

[9] Knuth, Donald E., The Art of Computer Programming, v. 2, pp. 293 to 395, Addison-Wesley Publishing Co., Reading, Mass., 1969.

[10] Beltrami, E.J., Mathematics for Dynamic Modeling, pp. 217 to 218, Academic Press, Boston, Mass., 1987.

[11] Beaumont,R.A., "Polynomial Systems of Equations", McGraw-Hill Encyclopedia of Science and Technology, Vol. 14, pp. 151 to 152, McGraw-Hill Book Company, New York, 1987.

DISTRIBUTION LIST

REPORT NO. NADC-91112-50

	No. of Copies
Defense Technical Information Center ...... Cameron Station Alexandria, VA 22394	2
Office of the Chief of Naval Research .... Attn: Dr. William Lukens (Code 10P6) 800 N. Quincy St. Arlington, VA 22217-5000	1
NAVAIRWARCENACDIVWAR:	
Scientific and Technical Library,Code 8131	2
Code 01B	1
Code 301	1
Code 3011	1
Code 302	1
Code 3021	1
Code 303	1
Code 3031	1
Code 40F	1
Code 401	1
Code 402	1
Code 403	1
Code 404	1
Code 50	1
Code 501	1
Code 5012 (Dr. Jon Davis)	1
Code 5012 (Dr. Lloyd Bobb)	1
Code 502	1
Code 503	1
Code 5032 (Dr. Mary Eileen Farrell)	1
Code 5032 (Richard Wayland)	1
Code 504	1
Code 505	1
Code 505A (Dr. Robert M. Williams)	30
Code 5051	1
Code 5032 (Anthony Passamante).	1
Code 6012	1
Code 6051 (Dr. Richard Llorens)	1